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On a Probability Model of the Number of Conceptions Classified by the Nature of Terminations Based on a Bivariate Geiger Muller (G. M.) Counter

1. Introduction

A theoretical basis for obtaining the probability of a particular type of pregnancy terminations as live births, still births or foetal wastages classified by the order of pregnancy is a useful item of investigation in population Analysis. The data relating to the sequence of terminations classified by their nature are useful in evaluating the fertility performance of a community in terms of the impact of reproductive wastages on total fertility. Sovani and Dandekar (1955), Biswas (1964), etc. have shown that the interpregnancy intervals for all order of conceptions are substantially reduced following the termination of the previous pregnancy into reproductive wastage or still birth. This suggests the need to identify the biological factors influencing the nature of pregnancy terminations and then to build up a suitable probability model for predicting the sequence of pregnancy terminations classified by the nature of terminations.

In the present paper attempts have been made to evolve probability model which describes the intensity with which the mothers are exposed to the risk of pregnancies which is generally observed to vary with age, parity or marital exposure. Casual factors leading to different nature of pregnancy terminations into live birth, still birth on foetal wastages have been identified in terms of the occurrence and persistence of some kind of intra-uterine shocks during a particular phase of the gestation period assuming a certain time independent intensity, for the sake of simplicity.

The entire process of reproductive behaviour is thus modelled by a pair of Geiger Muller Counters. A Geiger Muller counter (G.M. Counter) is ordinarily a device which is used to record the arrival of radio active particles subject to the condition that any arrival is followed by a dead time during which the counter remains paralysed; either for a fixed or random variable dead time and thus fails to record the arrival of further particles during the dead time.

The first counter is thus hypothetically imagined to record the conceptions followed by fixed dead time with time dependent decreasing intensities. The second counter is assumed to record hypothetically maternal shocks occurring with fixed time independent intensity; which when occurring during the gestation period may affect the course of pregnancies as live birth, still birth or foetal wastages.

The problem comprises of obtaining a probability model of n number of conceptions classified by the nature of terminations arranged in different sequences of the outcomes of conceptions at times $x_1 < x_2 < x_3 \ldots < x_n$ measured from time origin $T = 0$, which is the date of effective marriage.

Donating by 'L' a live birth and 'S' a still birth or a foetal wastage, we have for three conceptions occurring at times $x_1 < x_2 <, x_3,$ a number of 2^3 sequence of possible outcome as follows:

TABLE 1 —SEQUENCE OF POSSIBLE OUTCOMES OF THREE CONCEPTIONS OCCURING AT (X_1 X_2 X_3)

Events	Times of Conceptions		
	I	II	III
	x_1	X_2	X_3
E_1	L	L	L
E_2	S	S	S
E_3	L	S	L
E_4	S	L	S
E_5	S	S	L
E_6	L	L	S
E_7	S	L	L
E_8	L	S	S

2. Assumptions for the Development of the Model

(i) We assume two series of events recorded in two Geiger Muller (G.M.) Counters.

(ii) The first counter refers to Geiger Muller (G.M.) counter type I with fixed dead time. This counter records conceptions at times $x_1 < x_2 < x_3 \ldots$ and following each conception there is a fixed dead time π during which no further conception can occur. In fact π is the duration of infecundable exposure following a conception; which for the sake of simplicity is assumed to be fixed and independent of the nature of conceptions. The intensity or the hazard rate of the conceptions occuring at time $T = x$ have been taken to be distributed as Weibull, given by

$$\phi_1(x) = \lambda_0 x^{\alpha-1}; \quad 0 < \alpha < 1, \quad 0 \leqslant \lambda_0 < \infty.$$

Note that for $0 < \alpha < 1$, the hazard rate is a decreasing function with increase in the maternal age x.

(iii) The second counter refers to Geiger Muller (G.M.) counter Type I with random variable dead time (Pyke Ronald (1969)). This counter records intrauterineshocks terminating a conception into foetal wastage or still birth which happens during the gestation period following a conception during a subinterval $(x_i, x_i + \pi')$; the latter belongs to a bigger interval $(x_i, x_i + \pi)$ $(i = 1, 2, 3)$ during which the first counter remains locked or paralysed.

(iv) Following each shock in the second counter we assume the presence of a dead time during which no further shocks can arrive. The dead time is considered to be a random variable (r.v.) with the waiting time distribution which is negative exponential with parameter μ.

The hazard function of the shock in the second counter is also taken as time independent constant λ leading to the inter-arrival distribution of shocks again to be negative exponential with parameter λ.

3. Development of the Probability Model

We note that the probability of three conceptions occurring at times $x_1 < x_2 < x_3$ measured from the time origin $T = 0$ (the date of effective marriage) subject to the condition that each conception is followed by a fixed dead time π (period of post partum infecundable ecposure) is given by the density function:

$$f(x_1, x_2, x_3 \mid \pi, \pi, \pi) = \phi_1(x_1)\,\phi_1(x_2)\,f_1(x_3) \left\{ \prod_{i=1}^{3} \left[1 - (F_1(x_i + \pi) - F_1(x_i)) \right] \right\}^{-1} \quad \text{(i)}$$

(for proof vide 'Appendix' of the paper)

where $\phi_1(x_i)$, $f_1(x_i)$and $F_1(x_i)$ denote respectively the hazard function, interval density and cumulative distribution function (c.d.f.) of the waiting time distribution of conception recorded at x_i $(i = 1, 2, 3)$.

In the Second place, we note that the probability that the second counter is open at T = t given that it was open at T — 0 is given by

$$\alpha(t) = 1 - F_2(t) + \int_0^t \phi_2(u)\,[1 - G(t-u)]\,du \qquad \text{(ii)}$$

where $F_2(t)$ represents the c.d.f. of the arrival time of a shock; $G(t)$ represents the survival function of the dead time following a shock and $\phi_2(u)$ represents the hazard rate of a shock at $T = u$. With these notations the probability of the events E_i ($i = 1, 2, \ldots, 8$) (defined in Table 1) are given in Appendix A_2

The demographic significance of the bivariate counter lies in recording (vis-a-vis the first counter) the time of conception during the free period of the counter (i.e. during the fecundable exposure) with gradually decreasing intensities with the age. While the second counter deciding the nature of terminations (i.e. live birth, still birth or foetal death) depending on the intrauterine shocks arriving during different phases of the gestation period with time independent poisson intensity. The rationale of time independent intensity rests on the premises that the maternal shocks may arrive at random in a memory less order irrespective of the age or parity of the mother, for the sake of simplicity.

4. Estimation of the Parameters of the Model

We propose to obtain the estimates of the parameters of the model viz. ($\lambda_0, \alpha, \lambda, \mu$) assumng our prior knowledge on the values of π and π', where π is the total infecundable exposure following a conception and π' corresponds to some initial part of π and arrival of shocks in the second counter during (x_i, $x_i + \pi'$) leads to still birth or foetal wastage.

A. In the first place (λ_0, α) the parameters involved in the occurrence of three conceptions for all the j women ($j = 1, 2, \ldots, n$) at times

$$x_{1j} < x_{2j} < x_{3j}$$

will be estimated by the method of maximum likelihood using L' which represents the joint probability density of three conceptions at times $x_{1j} < x_{2j} < x_{3j}$ ($j = 1, 2, \ldots n$) subject to the condition that each conception is followed by a fixed dead time π. The details of estimation technique are given in Appendix A_3.

B. The estimates of the parameters involved in the process of the second counter have been obtained by Neyman's method of modified minimum chi-square (χ^2) as follows :

The joint probaility densities of E_1 and E_2 are given by:

$$({}^{(1)}L'') = \prod_{j=1}^{n} \prod_{i=1}^{3} \Big[e^{-\lambda(x_{ij}+\pi')} - e^{-\lambda x_{ij}} - \frac{\lambda^2}{2} \Big\{ (x_{ij} + \pi')^2 - x_{ij}^2 \Big\}$$

$$+ \frac{\lambda^2}{\mu^2} \Big\{ e^{-\mu(x_{ij}+\pi')} - e^{-\mu x_{ij}} \Big\} \Big] + \pi'^n + \left(\frac{\lambda^2 \pi'}{\mu} \right)^n \qquad \text{(iii)}$$

and

$$({}^{(2)}L'') = \prod_{j=1}^{n} \prod_{i=1}^{3} \Big[e^{-\lambda x_{ij}} - e^{-\lambda(x_{ij}+\pi')} + \frac{\lambda^2}{2} \Big\{ (x_{ij} + \pi')^2 - x_{ij}^2 \Big\}$$

$$+ \frac{\lambda^2}{\mu^2} \Big[e^{-\mu x_{ij}} - e^{-\mu(x_{ij}+\pi')} \Big\} \Big] - \left(\frac{\lambda^2 \pi'}{\mu} \right)^n \qquad \text{(iv)}$$

One simple way is to replace the right hand side of (iii) and (iv) by their frequency estimates (relative frequencies) and the two equations involving λ and μ may be solved by successive iterations. But the estimates differ if we take instead of (${}^{(1)} L''$ and) ${}^{(2)} L''$) any two other events out of $\binom{8}{2} = 28$ events. But in any case (iii) and (iv) take account of the entire information contained in the data, of course, in a different way than those contained by any two other estimating equations.

Neyman's modified χ^2 technique applied on the observed and expected frequencies may provide consistently asymptotically normal (C.A.N.) estimates by minimizing

$$\chi^2 = Q^+ = \sum_{i=1}^{2} \frac{(N_i - NP^{+ 2})}{N_i} \qquad \text{(v)}$$

where P_i^+ is the linearized approximate of $P_i\,(i=1, 2)$ as given in (v).

Let the solution of (iii) and (iv) be given as $\bar{\lambda}$ and $\bar{\mu}$. Then the linearized approximation of

$$P_i = P(E_i)$$

is given by

$$P_i^+ \pm P_i(\bar{\lambda}, \bar{\mu}) + (\lambda - \bar{\lambda}) \frac{\partial P_i}{\partial \lambda}\Big|_{\bar{\lambda}, \bar{\mu}} + (\mu - \bar{\mu}) \frac{\partial P_i}{\partial \mu}\Big|_{\bar{\lambda}, \bar{\mu}} \qquad \text{(vi)}$$

where P_i's ($i = 1, 2$) are given in equations (xix) and (xx).

The C.A.N. estimates of (λ, μ) are obtained from (v) by equating to zero the partial derivatives of Q^+ with respect to λ and μ after substituting the linearized estimates for P_i^+ ($i = 1, 2$)

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APPENDIX

A_1

We have

$$F(x_1, x_2) = P_r(X_1 < x_1, X_2 < x_2)$$

$$= \int_0^{x_1} \frac{F(x_2) - F(x)}{1 - F(x)} \, dF(x)$$

$$= \int_0^{x_1} \frac{F(x_2)}{\bar{F}(x)} \, dF(x) - \int_0^{x_1} \frac{F(x)}{\bar{F}(x)} \, dF(x)$$

where $\bar{F}(x) = 1 - F(x)$

$$= F(x_2) \int_0^{x_1} \frac{dF(x)}{\bar{F}(x)} + [F(x_1) + \log_e(1 - F(x_1))]$$

$$= F(x_2) \int_0^{x_1} \phi(x)dx + [F(x_1) + \log_e(1 - F(x_1))]$$

where $\phi(x)$ is the hazard rate at $X = x$

$$= F(x_2)\, \Phi(x_1) + F(x_1) - \int_0^{x_1} \phi(x)dx$$

$$(\because \log(1 - F(x_1) = \log \bar{F}(x_1) = \log \left[e^{-\int_0^{x_1} \phi(x)dx} \right] = \int_0^{x_1} -\phi(x)dx)$$

where $\Phi(x_1) = \int_0^{x_1} \phi(x)dx$. Hence

$$F(x_1, x_2) = F(x_2)\phi(x_1) + F(x_1) - \Phi(x_1)$$

$$f(x_1, x_2) = \frac{\partial^2 F(x_1, x_2)}{\partial x_1 \, \partial x_2} = f(x_2)\phi(x_1).$$

Similarly one can show

$$f(x_1, x_2, x_3) = (\phi(x_1)\phi(x_2)f(x_3))$$

$$f(x_1, x_2, x_3, \ldots\ldots x_n) = \phi(x_1)\phi(x_2) \ldots\ldots \phi(x_{n-1})\, f(x_n).$$

Finally the conditional probability of having n conceptions at $x_1 < x_2 < \ldots\ldots$

< X_n subject to the condition that no events can occur between $(X_i, X_i + \pi)$, i = 1, 2, ... , n is given by

$$\frac{\phi(x_1)\phi(x_2)\ldots\phi(x_{n-1})f(x_n)}{\prod_{i=1}^{n}(1-(F(x_i+\pi)-F(x_i)))}$$

and for $n = 3$ the result (i) follows immediately.

A₂. The probabilities of the events Ei (defined in Table 1) are as follows:

$$P_1 = P(E_1) = f(x_1, x_2, x_3 \mid \pi, \pi, \pi) \prod_{i=1}^{3} \int_{x_i}^{x_i+\pi'} [1-\alpha(t)+\alpha(t)(1-\phi_2(t))]\,dt \tag{vii}$$

$$P_2 = P(E_2) = f(x_1, x_2, x_3 \mid \pi, \pi, \pi) \prod_{i=1}^{3} \int_{x_i}^{x_i+\Pi'} \alpha(t)\,\phi_2(t)\,dt \tag{viii}$$

$$P_3 = P(E_3) = f(x_1, x_2, x_3 \mid \pi, \pi, \pi) \prod_{i=1,3} \left\{ \int_{x_i}^{x_i+\pi'} [1-\alpha(t)+\alpha(t)(1-\phi_2(t))]\,dt \right\} \int_{x_2}^{x_2+\pi'} \alpha(t)\phi_2(t)\,dt \tag{ix}$$

$$P_4 = P(E_4) = f(x_1, x_2, x_3 \mid \pi, \pi, \pi) \prod_{i=1,3} \left\{ \int_{x_i}^{x_i+\pi'} \alpha(t)\phi_2(t)dt \right\} \int_{x_2}^{x_2+\pi'} [1-\alpha(t)+\alpha(t)(1-\phi_2(t))]\,dt \tag{x}$$

$$P_5 = P(E_5) = f(x_1, x_2, x_3 \mid \pi, \pi, \pi) \prod_{i=1}^{2} \left\{ \int_{x_i}^{x_i+\pi'} \alpha(t)\phi_2(t)dt \right\} \int_{x_3}^{x_3+\pi'} [1-\alpha(t)+\alpha(t)(1-\phi_2(t))]\,dt \tag{xi}$$

$$P_6 = P(E_6) = f(x_1, x_2, x_3 \mid \pi, \pi, \pi) \prod_{i=1}^{2} \left\{ \int_{x_i}^{x_i+\pi'} [1-\alpha(t)+\alpha(t)(1-\phi_2(t))]\,dt \right\} \int_{x_3}^{x_3+\pi'} \alpha(t)\phi_2(t)dt \tag{xii}$$

$$P_7 = P(E_7) = f(x_1, x_2, x_3 \mid \pi, \pi, \pi) \int_{x_1}^{x_1+\pi'} \alpha(t)\phi_2(t)dt$$

$$\prod_{i=2}^{3} \left\{ \int_{x_i}^{x_i+\pi'} [1 - \alpha(t) + \alpha(t)(1 - \phi_2(t))]\, dt \right\} \qquad \text{(xiii)}$$

$$P_8 = P(E_8) = f(x_1, x_2, x_3 \mid \pi, \pi, \pi) \int_{x_1}^{x_1+\pi'} [1 - \alpha(t) + \alpha(t)\,(1 - \phi_2(t))]\, dt$$

$$\prod_{i=2}^{3} \left\{ \int_{x_i}^{x_1+\pi'} \alpha(t)\phi_2(t)\, dt \right\} \qquad \text{(xiv)}$$

Admitting

$$\phi_1(x_i) = \lambda_0\, x_i^{\alpha-1}$$

$$f_1(x_i) = \lambda_0\, x_i^{\alpha-1}\, e^{-(\lambda_0/\alpha)\, x_i^{\alpha}};\ 0 < \lambda_0 < \infty,\ 0 < \alpha < 1$$

$$F_1(x_i) = 1 - e^{-(\lambda_0/\alpha) x_i^{\alpha}} \qquad \text{(xv)}$$

$$\phi_2(u) = \lambda; \quad 0 \leqslant \lambda < \infty, \quad t \geqslant 0$$

$$f_2(t) = \lambda e^{-\lambda t}$$

$$F_2(t) = 1 - e^{-\lambda t} \qquad \text{(xvi)}$$

$$G(t) = e^{-\mu t} \qquad \text{(xvii)}$$

$$\alpha(t) = e^{-\lambda t} + \int_0^t \lambda\, [1 - e^{-\mu(t-u)}]\, du$$

$$= e^{-\lambda t} + \lambda t - \frac{\lambda}{\mu}\,(1 - e^{-\mu t}) \qquad \text{(xviii)}$$

it follows that

$P_1 = P(E_1)$

$$= \lambda_0^3 (x_1 x_2 x_3)^{\alpha-1} e^{-(\lambda_0/\alpha) x_3^{\alpha}} \left\{ \prod_{i=1}^{3} \left[1 + e^{-(\lambda_0/\alpha)(x_i+\pi)^{\alpha}} - e^{-(\lambda_0/\alpha) x_i^{\alpha}} \right] \right\}^{-1}$$

$$\prod_{i=1}^{3} \left[\pi' + e^{-\lambda(x_i+\pi')} - e^{-\lambda x_i} - \frac{\lambda^2}{2} \left\{ (x_i + \pi')^2 - x_i^2 \right\} \right.$$

$$\left. + \frac{\lambda^2}{\mu}\, \pi' + \frac{\lambda^2}{\mu^2} \left\{ e^{-\mu(x_i+\pi')} - e^{-\mu x_i} \right\} \right] \qquad \text{(xix)}$$

Similarly,

$P_2 = P(E_2)$

$$= \lambda_0^3 (x_1 x_2 x_3)^{\alpha-1} e^{-(\lambda_0/\alpha) x_3^\alpha} \left\{ \prod_{i=1}^{3} \left[1 + e^{-\lambda_0/\alpha (x_i+\pi)^\alpha} - e^{-(\lambda_0/\alpha)\, x_i^\alpha} \right] \right\}^{-1}$$

$$\prod_{i=1}^{3} \left[e^{-\lambda x_i} - e^{-\lambda(x_i+\pi')} + \frac{\lambda^2}{2} \left\{ (x_i + \pi')^2 - x_i^2 \right\} \right.$$

$$\left. - \frac{\lambda^2}{\mu} \pi' + \frac{\lambda^2}{\mu^2} \left\{ e^{-\mu x_i} - e^{-\mu(x_i+\pi')} \right\} \right] \qquad \text{(xx)}$$

Writing

$$A_i = \pi' + e^{-\lambda(x_i+\pi')} - e^{-\lambda x_i} - \frac{\lambda^2}{2} \left\{ (x_i + \pi')^2 - x_i^2 \right\}$$

$$+ \frac{\lambda^2}{\mu} \pi' + \frac{\lambda^2}{\mu^2} \left\{ e^{-\mu(x_i+\pi')} - e^{-\mu x_i} \right\}$$

$$B_i = e^{-\lambda x_i} - e^{-\lambda(x_i+\pi')} + \frac{\lambda^2}{2} \left\{ (x_i + \pi')^2 - x_i^2 \right\}$$

$$- \frac{\lambda^2}{\mu} \pi' + \frac{\lambda^2}{\mu^2} \left\{ e^{-\mu x_i} - e^{-\mu(x_i+\pi')} \right\}$$

and

$$\Delta = \lambda_0^3 (x_1\ x_2\ x_3)^{\alpha-1} e^{-(\lambda_0/\alpha) x_3^\alpha} \left\{ \prod_{i=1}^{3} \left[1 + e^{-(\lambda_0/\alpha)(x_i+\pi)^\alpha} - e^{-(\lambda_0/\alpha)\, x_i^\alpha} \right] \right\}^{-1} \qquad \text{(xxi)}$$

We can write the probabilities of the other events as

$$\begin{aligned} P_3 &= P(E_3) = \Delta\ A_1\ B_2\ A_3 \\ P_4 &= P(E_4) = \Delta\ B_1\ A_2\ B_3 \\ P_5 &= P(E_5) = \Delta\ B_1\ B_2\ A_3 \\ P_6 &= P(E_6) = \Delta\ A_1\ A_2\ B_3 \\ P_7 &= P(E_7) = \Delta\ B_1\ A_2\ B_3 \\ P_8 &= P(E_8) = \Delta\ A_1\ B_2\ B_3 \end{aligned} \qquad \text{(xxii)}$$

A_3. The parameters (λ_0, α) will be estimated by method of maximum likelihood as follows :

L' = Joint probability density of three conceptions at times $x_{1j} < x_{2j} < x_{3j}$ $(J = 1, 2, \ldots, n)$ subject to the condition that each conception is followed by a fixed dead time π.

$$L' = \prod_{j=1}^{n} \left[\lambda_0^3 \left(x_{1j}\, x_{2j}\, x_{3j} \right)^{\alpha-1} e^{-(\lambda_0/\alpha)\, x_{3j}^{\alpha}} \right]$$

$$\left[\prod_{j=1}^{n} \prod_{i=1}^{3} \left[1 + e^{-(\lambda_0/\alpha)\, (x_{ij}+\pi)^{\alpha}} - e^{-(\lambda_0/\alpha)\, x_{ij}^{\alpha}} \right] \right\}^{-1} \qquad \textbf{(xxiii)}$$

$$\Rightarrow \log_j L' = \sum_{j=1}^{n} \log \left[\lambda_0^3 \left(x_{1j}\, x_{2j}\, x_{3j} \right)^{\alpha-1} e^{-(\lambda_0/\alpha)\, x_{3j}^{\alpha}} \right]$$

$$- \sum_{j=1}^{n} \sum_{j=1}^{3} \log \left[1 + e^{-(\lambda_0/\alpha)\, (x_{ij}+\pi)^{\alpha}} - e^{-(\lambda_0/\alpha)\, x_{ij}^{\alpha}} \right]$$

$$= 3n \log \lambda_0 + (\alpha - 1) \sum_{i=1}^{3} \sum_{j=1}^{n} \log x_{ij} - \frac{\lambda_0}{\alpha} \sum_{j=1}^{n} x_{3j}^{\alpha}$$

$$- \sum_{=1}^{n} \sum_{i=1}^{3} \log \left[1 + e^{-(\lambda_0/\alpha)\, (x_{ij}+\pi)^{\alpha}} - e^{-(\lambda_0/\alpha) x_{ij}^{\alpha}} \right]$$

Setting

$$\frac{\partial \log L'}{\partial \lambda_0} = 0 \Rightarrow$$

$$\frac{3n}{\lambda_0} - \frac{1}{\alpha} \sum_{j=1}^{n} x_{3j}^{\alpha} + \Bigg[\sum_{j=1}^{n} \sum_{i=1}^{3} \Bigg\{ \frac{(x_{ij}+\pi)^{\alpha}}{\alpha}\, e^{-(\lambda_0/\alpha)\, (x_{ij}+\pi)^{\alpha}}$$

$$- \frac{x_{ij}^{\alpha}}{\alpha} e^{-(\lambda_0/\alpha)\, x_{ij}^{\alpha}} \Bigg\} \left\{ 1 + e^{-(\lambda_0/\alpha)\, (x_{ij}+\pi)} - e^{-(\lambda_0/\alpha)\, x_{ij}^{\alpha}} \right\}^{-1} \Bigg] = 0 \qquad \textbf{(xxiv)}$$

and

$$\frac{\partial \log L'}{\partial \alpha} = 0 \Rightarrow$$

$$\sum_{j=1}^{n} \log (x_{1j} x_{2j} x_{3j}) + \frac{\lambda_0}{\alpha^2} \sum_{j=1}^{n} x_{3j}^{\alpha} - \frac{\lambda_0}{\alpha} \sum_{j=1}^{n} x_{3j}^{\alpha} \log x_{3j}$$

$$- \sum_{j=1}^{n} \sum_{i=1}^{3} \Bigg\{ \frac{\lambda_0}{\alpha} (x_{ij}+\pi)^{\alpha}\, e^{-(\lambda_0/\alpha)\, (x_{ij}+\pi)^{\alpha}} \left[\frac{1}{\alpha} - \log (x_{ij}+\pi) \right]$$

$$- \frac{\lambda_0}{\alpha} x_{ij}^{\alpha}\, e^{-(\lambda_0/\alpha)\, (x_{ij}^{\alpha}} \left[\frac{1}{\alpha} - \log x_{ij} \right] \Bigg\}$$

$$\left\{ 1 + e^{-(\lambda_0/\alpha)(x_{ij}+\pi)^\alpha} - e^{-(\lambda_0/\alpha) x_{ij}^\alpha} \right\}^{-1} = 0 \qquad \text{(xxv)}$$

It is difficult to get direct solution of (λ_0, α) from the estimating likelihood equations (xxiv) and (xxv).

However if we start with an approximate estimate of (λ_0, α) say $(\lambda_0^{(0)}, \alpha^{(0)})$ then by successive iterations one can get the maximum likelihood estimates of λ_0 and α. To get a knowledge of the starting value of $(\lambda_0^{(0)}, \alpha^{(0)})$ one may use the data relating to the time of the first or any other order of conception only. From the sample estimates of the mean and variance of the waiting time of first conception one may get, by using

$$\hat{\mu} = \text{Sample Mean}$$

$$= \Gamma\left(1 + \frac{1}{\alpha^{(0)}}\right) \Big/ \left(\frac{\lambda_0^{(0)}}{\alpha^{(0)}}\right) \qquad \text{(xxvi)}$$

and $\hat{\sigma}^2$ = Sample Variance

$$= \frac{\left(1 + \frac{2}{\alpha^{(0)}}\right) - \left[\Gamma\left(1 + \frac{1}{\alpha^{(0)}}\right)\right]^2}{\left(\frac{\lambda^{(0)}}{\alpha^{(0)}}\right)^2 \Big/ \alpha^{(0)}} \qquad \text{(xxvii)}$$

The solutions of (xxvi) and (xxvii) give $(\lambda_0^{(0)}, \alpha^{(0)})$, which when applied in equations (xxvi) and (xxvii) will yield maximum likelihood estimates of λ_0 and α by successive iterations.